

STARLIKENESS OF THE GENERALIZED INTEGRAL TRANSFORM USING DUALITY TECHNIQUES

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ABSTRACT. For $\alpha \geq 0$, $\delta > 0$, $\beta < 1$ and $\gamma \geq 0$, the class $\mathcal{W}_\beta^\delta(\alpha, \gamma)$ consist of analytic and normalized functions f along with the condition

$$\operatorname{Re} e^{i\phi} \left((1-\alpha+2\gamma)(f/z)^\delta + \left(\alpha-3\gamma+\gamma \left[(1-1/\delta)(zf'/f) + 1/\delta(1+zf''/f') \right] \right) (f/z)^\delta (zf'/f) - \beta \right) > 0,$$

where $\phi \in \mathbb{R}$ and $|z| < 1$, is taken into consideration. The class $\mathcal{S}_s^*(\zeta)$ be the subclass of the univalent functions, defined by the analytic characterization $\operatorname{Re}(zf'/f) > \zeta$, for $0 \leq \zeta < 1$, $0 < \delta \leq \frac{1}{(1-\zeta)}$ and $|z| < 1$. The admissible and sufficient conditions on $\lambda(t)$ are examined, so that the generalized and non-linear integral transforms

$$V_\lambda^\delta(f)(z) = \left(\int_0^1 \lambda(t) (f(tz)/t)^\delta dt \right)^{1/\delta},$$

maps the function from $\mathcal{W}_\beta^\delta(\alpha, \gamma)$ into $\mathcal{S}_s^*(\zeta)$. Moreover, several interesting applications for specific choices of $\lambda(t)$ are discussed, that are related to some well-known integral operators.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and \mathcal{A} be the class of all normalized and analytic functions f defined in the domain \mathbb{D} with the condition $f(0)=0=f'(0)-1$. Further, let $\mathcal{S} \subset \mathcal{A}$ denote the class consisting of all univalent functions in \mathbb{D} .

In [13], R. Fournier and S.Ruscheweyh considered the the linear functional

$$L_\Lambda(f) = \inf_{z \in \mathbb{D}} \int_0^1 \Lambda(t) \left(\operatorname{Re} \frac{f(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt, \quad f \in \mathcal{S},$$

and

$$L_\Lambda(\mathcal{S}) := \inf_{f \in \mathcal{S}} L_\Lambda(f)$$

for an integrable function $\Lambda : [0, 1] \rightarrow \mathbb{R}$, positive in $(0, 1)$. Since $L_\Lambda(F) \leq 0$ for the Koebe function F , it is clear that $L_\Lambda(\mathcal{S}) \leq 0$ for every admissible weight function Λ and R. Fournier and S.Ruscheweyh [13] posed the problem of finding existence and characterization of the weight functions for which $L_\Lambda(\mathcal{S}) = 0$. As mentioned in [13], since it is not possible to solve the problem for the class \mathcal{S} , the main focus was shifted into its important subclass of close-to-convex functions \mathcal{C} . For further study of this problem, the following subclass of \mathcal{S} is important.

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The class $\mathcal{S}^*(\xi)$ having the analytic characterization

$$\operatorname{Re} \left(\frac{zf'}{f} \right) > \xi, \quad 0 \leq \xi < 1, \quad z \in \mathbb{D},$$

is the generalization of the class of starlike functions, $\mathcal{S}^* := \mathcal{S}^*(0)$, which contains all the functions f with the property that the domain $f(\mathbb{D})$ is starlike with respect to the origin.

Using the duality techniques, in [13], R. Fournier and S. Ruscheweyh provided solution to two related extremal problems as

- (i) For Λ integrable on $[0, 1]$ and positive on $(0, 1)$, if $\frac{\Lambda(t)}{1-t^2}$ is decreasing on $(0, 1)$, then $L_\Lambda(\mathcal{C}) = 0$, and
- (ii) $V_\lambda(\mathcal{P}_\beta) \subset \mathcal{S}^* \Leftrightarrow L_\Lambda(\mathcal{C}) = 0$, where

$$V_\lambda(f) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \quad (1.1)$$

with $\lambda : [0, 1] \rightarrow \mathbb{R}$ is nonnegative, satisfying $\int_0^1 \lambda(t) dt = 1$ and

$$\mathcal{P}_\beta = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(e^{i\alpha} (f'(z) - \beta) \right) > 0, \quad \alpha \in \mathbb{R}, \quad z \in \mathbb{D} \right\}.$$

Note that the operator (1.1) contains several well-known operators such as Bernardi, Komatu and Hohlov as its special cases for specific choices of $\lambda(t)$ and has been studied extensively by several authors. For details on these operators see [2, 7, 8] and references therein. Further study of this problem, where the operator (1.1) carries generalization of the functional class \mathcal{P}_β involving linear combinations of the functionals $f(z)/z$, $f'(z)$ and $zf''(z)$ to $\mathcal{S}^*(\xi)$ were carried by many researchers in the recent future. All these combinations were unified in a class introduced in [1] and for the most general result in this direction, see [8, 16, 18]. Since such combinations and the generalization of the operator encompass large number of previous results such as univalence, subordination and positivity results on classes of functions, we are interested in considering the following generalized operator

$$F_\delta(z) := V_\lambda^\delta(f)(z) = \left(\int_0^1 \lambda(t) \left(\frac{f(tz)}{t} \right)^\delta dt \right)^{1/\delta}, \quad \delta > 0. \quad (1.2)$$

The integral operator defined in (1.2) and its more generalized form was considered in the work of I. E. Bazilevič [4] (see also [12, 19]). Note that when $\delta = 1$, the operator (1.2) reduces to (1.1). Study of this operator is really useful for the generalization of the functional class \mathcal{P}_β which is defined as follows.

$$\mathcal{W}_\beta^\delta(\alpha, \gamma) = \left\{ f \in \mathcal{A} : \operatorname{Re} e^{i\phi} \left((1-\alpha+2\gamma) \left(\frac{f}{z} \right)^\delta + \left(\alpha-3\gamma+\gamma \left[\left(1-\frac{1}{\delta} \right) \left(\frac{zf'}{f} \right) + \frac{1}{\delta} \left(1+\frac{zf''}{f'} \right) \right] \right) \left(\frac{f}{z} \right)^\delta \left(\frac{zf'}{f} \right) - \beta \right) > 0, \quad z \in \mathbb{D}, \quad \phi \in \mathbb{R} \right\}.$$

Here, $\alpha \geq 0$, $\beta < 1$, $\gamma \geq 0$ and $\phi \in \mathbb{R}$. Note that $\mathcal{W}_\beta^\delta(\alpha, 0) \equiv P_\alpha(\delta, \beta)$ is the class considered by A. Ebadian et al in [12], $R_\alpha(\delta, \beta) \equiv \mathcal{W}_\beta^\delta(\alpha + \delta + \delta\alpha, \delta\alpha)$ is a closely related class and $\mathcal{W}_\beta^1(\alpha, \gamma) \equiv \mathcal{W}_\beta(\alpha, \gamma)$ introduced by R.M. Ali et al in [1].

We also consider the following class related to $\mathcal{S}^*(\xi)$, given by

$$f \in \mathcal{S}_s^*(\zeta) \iff z^{1-\delta} f^\delta \in \mathcal{S}^*(\xi), \quad (1.3)$$

for $\xi = 1 - \delta + \delta\zeta$ and $0 \leq \xi < 1$. It is clear that the class $\mathcal{S}_s^*(\zeta)$ has the analytic characterization

$$\operatorname{Re} \left(\delta \frac{zf'}{f} + (1 - \delta) \right) > \xi, \quad \delta > 0, \quad 0 \leq \xi < 1.$$

Further, when $\delta = 1$, $\mathcal{S}_s^*(\zeta)$ and $\mathcal{S}^*(\xi)$ are equal. Wherever ξ is used in the sequel, it denotes the term $(1 - \delta + \delta\zeta)$.

In the present work the duality technique is used to determine the sharp estimate for the parameter β , so that the weighted integral operator V_λ^δ defined in (1.2) carries the function from $\mathcal{W}_\beta^\delta(\alpha, \gamma)$ to $\mathcal{S}_s^*(\zeta)$, where $0 \leq \zeta < 1$ and $0 < \delta \leq \frac{1}{(1-\zeta)}$. Certain related preliminaries and the main results involving the necessary and sufficient conditions are given in Section 2 which ensures $V_\lambda^\delta(\mathcal{W}_\beta^\delta(\alpha, \gamma)) \subset \mathcal{S}_s^*(\zeta)$. The simpler sufficient criterion are obtained in Section 3, which verifies $V_\lambda^\delta(f)(z) \in \mathcal{S}_s^*(\zeta)$, whenever $f \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$. Further, using these sufficient conditions, several interesting applications are studied for specific choices of $\lambda(t)$ are obtained in Section 4.

A closely related class $\mathcal{C}_\delta(\zeta)$ is defined as

$$\mathcal{C}_\delta(\zeta) := \left\{ f \in \mathcal{A} : (z^{2-\delta} f^{\delta-1} f') \in \mathcal{S}^*(\xi) \right\},$$

where $\xi := 1 - \delta + \delta\zeta$ with the conditions $1 - \frac{1}{\delta} \leq \zeta < 1$, $0 \leq \xi < 1$ and $\delta \geq 1$. In [9], similar analysis for $V_\lambda^\delta(f)(z) \in \mathcal{C}_\delta(\zeta)$, whenever $f \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$ are given. Various other inclusion properties, in particular, $V_\lambda^\delta(f)(z) \in \mathcal{W}_{\beta_1}^{\delta_1}(\alpha_1, \gamma_1)$, whenever $f \in \mathcal{W}_{\beta_2}^{\delta_2}(\alpha_2, \gamma_2)$ are given in [10].

2. PRELIMINARIES AND MAIN RESULTS

We need the following tools throughout this work.

The convolution or Hadamard product (denoted by $*$), of two functions $f_1 = (a_0 + a_1z + a_2z^2 + \dots)$ and $f_2 = (b_0 + b_1z + b_2z^2 + \dots)$ is given by

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Further, let c_i ($i = 0, 1, \dots, p$) and d_j ($j = 0, 1, \dots, q$) are complex parameters with $d_j \neq 0, -1, \dots$ and $p \leq q + 1$. Then for $z \in \mathbb{D}$, the function

$${}_pF_q(c_1, \dots, c_p; d_1, \dots, d_q; z) = \sum_{n=0}^{\infty} \frac{(c_1)_n \dots (c_p)_n}{(d_1)_n \dots (d_q)_n n!} z^n,$$

is called generalized hypergeometric function, which can also be represented as ${}_pF_q$. For $n \in \mathbb{N}$, $(\varepsilon)_n$ is the Pochhammer symbol or shifted factorial, which is defined as $(\varepsilon)_n = \varepsilon(\varepsilon + 1)_{n-1}$ and $(\varepsilon)_0 = 1$. In particular, ${}_2F_1$ is the well known Gaussian hypergeometric function.

The parameters $\mu, \nu \geq 0$ introduced in [1] are used for further analysis that are defined by the following relations

$$\mu\nu = \gamma \quad \text{and} \quad \mu + \nu = \alpha - \gamma. \quad (2.1)$$

Clearly (2.1) leads to two cases.

- (i) $\gamma = 0 \implies \mu = 0, \nu = \alpha \geq 0$.
- (ii) $\gamma > 0 \implies \mu > 0, \nu > 0$.

Define the auxiliary function

$$\psi_{\mu,\nu}^\delta(z) := \sum_{n=0}^{\infty} \frac{\delta^2 z^n}{(\delta + n\nu)(\delta + n\mu)} = \int_0^1 \int_0^1 \frac{1}{(1 - u^{\nu/\delta}/v^{\mu/\delta}z)} du dv. \quad (2.2)$$

Hence

$$\Phi_{\mu,\nu}^\delta(z) := (z\psi_{\mu,\nu}^\delta(z))' = \sum_{n=0}^{\infty} \frac{(n+1)\delta^2 z^n}{(\delta + n\nu)(\delta + n\mu)} = \frac{\delta^2}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{\delta/\nu-1}v^{\delta/\mu-1}}{(1 - uvz)^2} du dv. \quad (2.3)$$

Taking the case $\gamma = 0$ ($\mu = 0, \nu = \alpha \geq 0$), let $g_{0,\alpha}^\delta(t)$ be the solution of the differential equation

$$\frac{d}{dt} \left(t^{\delta/\alpha} \frac{(1 + g_{0,\alpha}^\delta(t))}{2} \right) = \frac{\delta(1-\xi)(1+t)}{\alpha(1-\xi)(1+t)^2} t^{\delta/\alpha-1}, \quad (2.4)$$

with the initial condition $g_\alpha^\delta(0) = 1$. By an easy calculation, the solution of (2.4) is given as

$$\frac{1 + g_{0,\alpha}^\delta(t)}{2} = \frac{\delta t^{-\delta/\alpha}}{\alpha} \int_0^t \frac{r^{\delta/\alpha-1}(1 - \xi(1+r))}{(1 - \xi)(1+r)^2} dr.$$

For the case $\gamma > 0$ ($\mu > 0, \nu > 0$), let $g_{\mu,\nu}^\delta(t)$ be the solution of the differential equation

$$\frac{d}{dt} \left(t^{\delta/\nu} \frac{(1 + g_{\mu,\nu}^\delta(t))}{2} \right) = \frac{\delta^2 t^{\delta/\nu-1}}{\mu\nu} \int_0^1 \frac{1 - \xi(1+st)}{(1 - \xi)(1+st)^2} s^{\delta/\mu-1} ds, \quad (2.5)$$

with the initial condition $g_{\mu,\nu}^\delta(0) = 1$. By an easy calculation, the solution of (2.5) can be given as

$$\frac{1 + g_{\mu,\nu}^\delta(t)}{2} = \int_0^1 \int_0^1 \frac{1 - \xi(1 + tr^{\nu/\delta}s^{\mu/\delta})}{(1 - \xi)(1 + tr^{\nu/\delta}s^{\mu/\delta})^2} dr ds. \quad (2.6)$$

Moreover for given $\lambda(t)$ and $\delta > 0$, we introduce

$$\Lambda_\nu^\delta(t) := \int_t^1 \frac{\lambda(s)}{s^{\delta/\nu}} ds, \quad \nu > 0, \quad (2.7)$$

and

$$\Pi_{\mu,\nu}^\delta(t) := \begin{cases} \int_t^1 \frac{\Lambda_\nu^\delta(s)}{s^{\delta/\mu-\delta/\nu+1}} ds & \gamma > 0 (\mu > 0, \nu > 0), \\ \Lambda_\alpha^\delta(t) & \gamma = 0 (\mu = 0, \nu = \alpha \geq 0). \end{cases} \quad (2.8)$$

These information, for $\delta = 1$ coincide with the one given in [18].

Our main aim is to establish both necessary and sufficient conditions that ensure $F_\delta(z) = V_\lambda^\delta(f)(z) \in \mathcal{S}_s^*(\zeta)$, whenever $f \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$. We state the conditions required for $F_\delta(z) = V_\lambda^\delta(\mathcal{W}_\beta^\delta(\alpha, \gamma))(z)$ to be in $\mathcal{S}_s^*(\zeta)$ and satisfy univalence in the following result and the proof of the same is given in Section 5.

Theorem 2.1. Let $\mu \geq 0$, $\nu \geq 0$ are defined in (2.1), $\delta \geq 1$ and $(1 - \frac{1}{\delta}) \leq \zeta \leq (1 - \frac{1}{2\delta})$. Let $\beta < 1$ satisfy

$$\frac{\beta}{(1 - \beta)} = - \int_0^1 \lambda(t) g_{\mu, \nu}^\delta(t) dt, \quad (2.9)$$

where $g_{\mu, \nu}^\delta(t)$ is defined by the differential equation (2.4) for $\gamma = 0$ and (2.5) for $\gamma > 0$. Assume that

$$\lim_{t \rightarrow 0^+} t^{\delta/\nu} \Lambda_\nu^\delta(t) \rightarrow 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} t^{\delta/\mu} \Pi_{\mu, \nu}^\delta(t) \rightarrow 0.$$

Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $F_\delta \in \mathcal{S}_s^*(\zeta)$ or $(z^{1-\delta}(F_\delta(z)))^\delta \in \mathcal{S}^*(\xi)$, where $\xi = 1 - \delta + \delta\zeta$ and $0 \leq \xi \leq 1/2$ if, and only if, $\mathcal{N}_{\Pi_{\mu, \nu}^\delta}(h_\xi) \geq 0$, where

$$\mathcal{N}_{\Pi_{\mu, \nu}^\delta}(h_\xi)(z) := \begin{cases} \int_0^1 t^{\delta/\mu-1} \Pi_{\mu, \nu}^\delta(t) \left(\operatorname{Re} \frac{h_\xi(tz)}{tz} - \frac{1 - \xi(1+t)}{(1-\xi)(1+t)^2} \right) dt, & \gamma > 0 (\mu > 0, \nu > 0), \\ \int_0^1 t^{\delta/\alpha-1} \Lambda_\alpha^\delta(t) \left(\operatorname{Re} \frac{h_\xi(tz)}{tz} - \frac{1 - \xi(1+t)}{(1-\xi)(1+t)^2} \right) dt, & \gamma = 0 (\mu = 0, \nu = \alpha \geq 0), \end{cases}$$

$$\text{and} \quad h_\xi(z) := z \left(\frac{1 + \frac{\epsilon+2\xi-1}{2(1-\xi)}z}{(1-z)^2} \right), \quad |\epsilon| = 1. \quad (2.10)$$

The value of β is sharp.

Remark 2.1. 1. For $\delta = 1$ and $\xi = 0$, Theorem 2.1 gives the result of [1, Theorem 3.1].
 2. For $\delta = 1$, Theorem 2.1 reduces to [18, Theorem 3.1] (see also [16, Theorem 2.1]).
 3. For $\gamma = 0$, Theorem 2.1 provides the result of [12, Theorem 2.1].

The condition equivalent to $\mathcal{N}_{\Pi_{\mu, \nu}^\delta}(h_\xi) \geq 0$ derived in Theorem 2.1 is provided in the following result which is useful for further discussion.

Theorem 2.2. Let $\gamma \geq 0$ ($\mu \geq 0$, $\nu \geq 0$), $\delta \geq 1$ and $(1 - \frac{1}{\delta}) \leq \zeta \leq (1 - \frac{1}{2\delta})$. Assume that the functions $\Lambda_\nu^\delta(t)$ and $\Pi_{\mu, \nu}^\delta(t)$, defined in (2.7) and (2.8), respectively are positive on $t \in (0, 1)$ and integrable on $t \in [0, 1]$. If $\beta < 1$ satisfy (2.9) and

$$\frac{t^{\delta/\mu-1} \Pi_{\mu, \nu}^\delta(t)}{(1+t)(1-t)^{3-2\delta(1-\zeta)}} \quad (2.11)$$

is decreasing on $t \in (0, 1)$. Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $F_\delta = V_\lambda^\delta(f)(z) \in \mathcal{S}_s^*(\zeta)$ or $z^{1-\delta}(F_\delta(z))^\delta \in \mathcal{S}^*(\xi)$, where $\xi = 1 - \delta + \delta\zeta$ and $0 \leq \xi \leq 1/2$.

Proof. For $t \in (0, 1)$, the mapping $t \rightarrow M(t)$ satisfies the condition that $M(t)$ is a positive function which decreases with respect to t and fits into the requirement leading to

$$\operatorname{Re} \int_0^1 M(t) \left(\frac{h_\xi(tz)}{tz} - \frac{1 - \xi(1+t)}{(1-\xi)(1+t)^2} \right) dt \geq 0$$

if, and only if, $F_1(z) = V_\lambda^1(f)(z) \in \mathcal{S}_s^*(\zeta)$, where $0 \leq \zeta \leq 1/2$ or $z^{1-\delta}(F_\delta(z))^\delta \in \mathcal{S}^*(\xi)$, where $\xi = 1 - \delta + \delta\zeta$, $\delta \geq 1$ and $0 \leq \xi \leq 1/2$, then it is enough to verify that

$$\frac{M(t)}{(1+t)(1-t)^{1+2\xi}}$$

is decreasing on $(0, 1)$. This is given already in [16, 18]. Further for $\xi = 0$ the same is proved in [1, 13]. Hence, if we take $M(t) = t^{\delta/\mu-1}\Pi_{\mu,\nu}^\delta(t)$, then we see that (2.11) satisfies the above observation and hence we get

$$\operatorname{Re} \int_0^1 t^{\delta/\mu-1} \Pi_{\mu,\nu}^\delta(t) \left(\frac{h_\xi(tz)}{tz} - \frac{1-\xi(1+t)}{(1-\xi)(1+t)^2} \right) dt \geq 0.$$

This proves the required result. \square

Remark 2.2. The condition (2.11) obtained in Theorem 2.2 cannot be reduced to the result obtained by A. Ebadian et al. [12, Theorem 2.2] for the case $\gamma = 0$. This is because the condition given in (2.11) contains the function $(1+t)(1-t)^{1+2\xi}$ whereas in [12], the result contains the function $(\log(1/t))^{1+2\xi}$ in its denominator part. Both the functions are decreasing and tends to 0 as $t \rightarrow 1$.

3. SUFFICIENT CRITERION OF THEOREM 2.2

In this section, the conditions are determined which ensures the sufficiency of Theorem 2.2 by a simpler method, so that the weighted integral operator $V_\lambda^\delta(\mathcal{W}_\beta^\delta(\alpha, \gamma)) \subset \mathcal{S}_s^*(\zeta)$, with $(1-\frac{1}{\delta}) \leq \zeta \leq (1-\frac{1}{2\delta})$ and $\delta \geq 1$. The conditions comprise of the following two cases.

Case (i). $\gamma > 0$ ($\mu > 0, \nu > 0$). In accordance of Theorem 2.2, the equivalent condition are obtained for the function

$$\frac{t^{\delta/\mu-1} \Pi_{\mu,\nu}^\delta(t)}{(1+t)(1-t)^{1+2\xi}},$$

which decreases in the range $t \in (0, 1)$, where $\Pi_{\mu,\nu}^\delta(t)$ is defined in (2.8), $\xi = 1 - \delta + \delta\zeta$, $\delta \geq 1$ and $\xi \in [0, 1/2]$.

Let

$$k(t) := \frac{l(t)}{(1+t)(1-t)^{1+2\xi}}, \quad \text{where } l(t) := t^{\delta/\mu-1} \Pi_{\mu,\nu}^\delta(t),$$

then taking logarithmic derivative of $k(t)$ will give

$$\frac{k'(t)}{k(t)} = \frac{l'(t)}{l(t)} + \frac{2(t+\xi+\xi t)}{(1-t^2)}.$$

For $t \in (0, 1)$, it is easy to note that $k(t) \geq 0$. Thus to prove that $k(t)$ is decreasing function of $t \in (0, 1)$ is equivalent of getting

$$p(t) := l(t) + \frac{(1-t^2)l'(t)}{2(t+\xi(1+t))} \leq 0.$$

Clearly $p(1) = 0$ means that if $p(t)$ is increasing function of $t \in (0, 1)$ then $k'(t) \leq 0$ and the proof is complete. Thus it is enough to show that $p'(t) \geq 0$, where

$$p'(t) = \frac{(1+t)}{2(t+\xi(1+t))^2} \left((-1-\xi+2\xi^2+t+3\xi t+2\xi^2 t)l'(t) + (1-t)(\xi+t+\xi t)l''(t) \right).$$

Differentiating $l(t)$ with respect t gives

$$\begin{aligned} l'(t) &= \left(\frac{\delta}{\mu} - 1 \right) t^{\delta/\mu-2} \Pi_{\mu,\nu}^\delta(t) - t^{\delta/\nu-2} \Lambda_\nu^\delta(t) \quad \text{and} \\ l''(t) &= \left(\frac{\delta}{\mu} - 1 \right) \left(\frac{\delta}{\mu} - 2 \right) t^{\frac{\delta}{\mu}-3} \Pi_{\mu,\nu}^\delta(t) - \left[\left(\frac{\delta}{\mu} + \frac{\delta}{\nu} - 3 \right) \right] t^{\frac{\delta}{\nu}-3} \Lambda_\nu^\delta(t) + t^{-2} \lambda(t). \end{aligned}$$

It is easy to see that the terms $(1+t)$ and $2(t+\xi(1+t))^2$ in the function $p'(t)$ are positive for $t \in (0, 1)$ and $\xi \in [0, 1/2]$. Now it only remains to show that

$$q(t) = \frac{Y(t)}{t} l'(t) + X(t) l''(t) \geq 0,$$

where

$$X(t) := (1-t)(t+\xi(1+t)) \quad \text{and} \quad Y(t) := -(1+2\xi)t(1-t-\xi(1+t)). \quad (3.1)$$

For $t = 1$, the function $q(t) = 0$. Therefore, if $q(t)$ is decreasing function of $t \in (0, 1)$ directly implies $k'(t) \leq 0$. Differentiating $q(t)$ with respect to t will give

$$\begin{aligned} q'(t) &= \left(\frac{\delta}{\mu} - 1 \right) t^{\delta/\mu-4} \Pi_{\mu,\nu}^\delta(t) \left(\left[tY'(t) + \left(\frac{\delta}{\mu} - 3 \right) Y(t) \right] + \left(\frac{\delta}{\mu} - 2 \right) \left[tX'(t) + \left(\frac{\delta}{\mu} - 3 \right) X(t) \right] \right) \\ &\quad - t^{\delta/\nu-4} \Lambda_\nu^\delta(t) \left(\left(\frac{\delta}{\mu} - 1 \right) \left[Y(t) + \left(\frac{\delta}{\mu} - 2 \right) X(t) \right] + tY'(t) + \left(\frac{\delta}{\mu} + \frac{\delta}{\nu} - 3 \right) tX'(t) \right. \\ &\quad \left. + \left(\frac{\delta}{\nu} - 3 \right) \left[Y(t) + \left(\frac{\delta}{\mu} + \frac{\delta}{\nu} - 3 \right) X(t) \right] \right) + t^{-3} \lambda(t) \left(Y(t) + \left(\frac{\delta}{\mu} + \frac{\delta}{\nu} - 5 \right) X(t) + tX'(t) \right) \\ &\quad + t^{-2} X(t) \lambda'(t). \end{aligned}$$

For $t \in (0, 1)$, the function $q'(t) \leq 0$ is counterpart of the following inequalities

$$\left(\frac{\delta}{\mu} - 1 \right) \left(\left[tY'(t) + \left(\frac{\delta}{\mu} - 3 \right) Y(t) \right] + \left(\frac{\delta}{\mu} - 2 \right) \left[tX'(t) + \left(\frac{\delta}{\mu} - 3 \right) X(t) \right] \right) \leq 0, \quad (3.2)$$

$$\begin{aligned} &\left(\frac{\delta}{\mu} - 1 \right) \left[Y(t) + \left(\frac{\delta}{\mu} - 2 \right) X(t) \right] + \left(\frac{\delta}{\nu} - 3 \right) Y(t) + tY'(t) \\ &\quad + \left(\frac{\delta}{\mu} + \frac{\delta}{\nu} - 3 \right) \left[\left(\frac{\delta}{\nu} - 3 \right) X(t) + tX'(t) \right] \geq 0 \end{aligned} \quad (3.3)$$

$$\text{and} \quad \lambda(t) \left(Y(t) + \left(\frac{\delta}{\mu} + \frac{\delta}{\nu} - 5 \right) X(t) + tX'(t) \right) + tX(t) \lambda'(t) \leq 0.$$

Letting $1 \leq \delta \leq \mu$ and $1 \leq \delta \leq \nu$, directly implies that the inequalities (3.2) and (3.3) are positive, which clearly means that the function $k(t)$ is decreasing on $t \in (0, 1)$ is equivalent of getting

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu} - \frac{(tX'(t) + Y(t))}{X(t)}, \quad (3.4)$$

where $1 \leq \delta \leq \mu$, $1 \leq \delta \leq \nu$, and $\xi \in [0, 1/2]$. Using (3.1), an easy calculation gives $tX'(t) + Y(t) \leq 0$, which clearly means that the inequality (3.4) is true when

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}, \quad 1 \leq \delta \leq \mu \quad \text{and} \quad 1 \leq \delta \leq \nu.$$

Summarizing these conditions, the general result for the case $\gamma > 0$ is stated as follows.

Theorem 3.1. *Let $\beta < 1$ satisfy (2.9) and let $\lambda(t)$ be real-valued, non-negative and integrable function for $t \in (0, 1)$. Further assume that the functions $\Lambda_\nu^\delta(t)$ and $\Pi_{\mu,\nu}^\delta(t)$ defined in (2.7) and (2.8), respectively are positive on $(0, 1)$ and integrable on $[0, 1]$. Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $F_\delta = V_\lambda^\delta(f)(z)$ belongs to the class $\mathcal{S}_s^*(\zeta)$ with $(1 - \frac{1}{\delta}) \leq \zeta \leq (1 - \frac{1}{2\delta})$ or $z^{1-\delta}(F_\delta(z))^\delta \in \mathcal{S}^*(\xi)$, where $\xi = 1 - \delta + \delta\zeta$, $\xi \in [0, 1/2]$, $\delta \geq 1$ and $\gamma > 0$ whenever*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}. \quad (3.5)$$

Case (ii). Let $\gamma = 0$ ($\mu = 0$, $\nu = \alpha \geq 0$). In accordance of Theorem 2.2 the equivalent condition are obtained for the function

$$a(t) := \frac{t^{\delta/\alpha-1}\Lambda_\alpha^\delta(t)}{(1+t)(1-t)^{1+2\xi}} = \frac{b(t)}{(1+t)(1-t)^{1+2\xi}}, \quad (3.6)$$

which decreases in the range $t \in (0, 1)$, where $\xi = (1 - \delta + \delta\zeta)$, $\xi \in [0, 1/2]$, $\delta \geq 1$ and $\Lambda_\alpha^\delta(t)$ is defined in (2.7).

For $\gamma = 0$, the subsequent two subcases are discussed for ξ .

At first, consider $\xi = 0$, then the function $a(t)$ corresponding to (3.6) is given as

$$a(t) := \frac{t^{\delta/\alpha-1}\Lambda_\alpha^\delta(t)}{(1-t^2)}.$$

Now, taking the logarithmic derivative of $a(t)$ will give

$$\frac{a'(t)}{a(t)} = \frac{2t}{(1-t^2)b(t)} \left(b(t) + \frac{(1-t^2)b'(t)}{2t} \right).$$

Thus to show that $a(t)$ is decreasing function of $t \in (0, 1)$ is equivalent of proving $c(t) := b(t) + (1-t^2)b'(t)/2t \leq 0$. For $t = 1$, $c(t) = 0$ which clearly implies that if $c'(t) \geq 0$ then the function $a(t)$ decreases and the proof is complete.

Now differentiating $c(t)$ gives

$$c'(t) = b'(t) + \frac{t(1-t^2)b''(t) - (1+t^2)b'(t)}{2t^2} = \frac{(1-t^2)}{2t^2} (tb''(t) - b'(t)),$$

where

$$b'(t) = \left(\frac{\delta}{\alpha} - 1\right) t^{\delta/\alpha-2} \Lambda_{\alpha}^{\delta}(t) - t^{-1} \lambda(t) \quad \text{and} \quad (3.7)$$

$$b''(t) = \left(\frac{\delta}{\alpha} - 1\right) \left(\frac{\delta}{\alpha} - 2\right) t^{\delta/\alpha-3} \Lambda_{\alpha}^{\delta}(t) - \left(\frac{\delta}{\alpha} - 2\right) t^{-2} \lambda(t) - t^{-1} \lambda'(t). \quad (3.8)$$

Thus $c'(t) \geq 0$ is equivalent of obtaining

$$\left(\frac{\delta}{\alpha} - 1\right) \left(\frac{\delta}{\alpha} - 3\right) \geq 0 \quad \text{and} \quad \frac{t\lambda'(t)}{\lambda(t)} \leq \left(3 - \frac{\delta}{\alpha}\right).$$

Thus for $\gamma \geq 0$ and $\xi = 0$, the following result can be stated.

Theorem 3.2. *Let $\beta < 1$ satisfy (2.9) and let $\lambda(t)$ be real-valued, non-negative and integrable function for $t \in (0, 1)$. Further assume that the functions $\Lambda_{\nu}^{\delta}(t)$ and $\Pi_{\mu, \nu}^{\delta}(t)$ defined in (2.7) and (2.8), respectively are positive on $(0, 1)$ and integrable on $[0, 1]$. Then for $f(z) \in \mathcal{W}_{\beta}^{\delta}(\alpha, \gamma)$, the function $F_{\delta} = V_{\lambda}^{\delta}(f)(z)$ belongs to the class $\mathcal{S}^*(1 - \frac{1}{\delta})$ or $z^{1-\delta}(F_{\delta}(z))^{\delta} \in \mathcal{S}^*$, $\delta \geq 1$, whenever*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq \begin{cases} 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}, & \gamma > 0 (\mu > 0, \nu > 0) \text{ and } 1 \leq \delta \leq \min\{\mu, \nu\}, \\ 3 - \frac{\delta}{\alpha}, & \gamma = 0 (\mu = 0, \nu = \alpha \in (0, \delta/3] \cup [\delta, \infty)). \end{cases}$$

Remark 3.1. *For $\delta = 1$, Theorem 3.2 provides better result for the case $\gamma > 0$ and similar result for the case $\gamma = 0$ when compared to [1, Theorem 4.2] and [18, Theorem 4.2] (see also [16, Theorem 2.3]). This is because, for the case $\delta = 1$ and $\gamma > 0$*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 5 - \frac{1}{\mu} - \frac{1}{\nu},$$

which gives 3 as the least value of right side term of the above expression. But in [1, Theorem 4.2], the bound is $1 + \frac{1}{\mu}$, where $\mu \geq 1$, which is clearly less than or equal to 2.

4. APPLICATIONS

In this section, using the conditions derived in Section 3, applications for various choices of $\lambda(t)$ are considered such that the conditions under which the generalized integral operator (1.2), for respective choice maps $\mathcal{W}_{\beta}^{\delta}(\alpha, \gamma)$ to $\mathcal{S}_s^*(\zeta)$ are examined.

To start with consider

$$\lambda(t) = (1 + c)t^c, \quad c > -1, \quad (4.1)$$

the integral operator (1.2) defined by the above weight function $\lambda(t)$ is known as generalized Bernardi operator denoted by (\mathcal{B}_c^{δ}) . This operator is the particular case of the generalized integral operators considered by [12] that follows in the sequel. For $\delta = 1$, this operator was introduced by S. D. Bernardi [5]. Now taking this operator the following result is obtained.

Theorem 4.1. *Let $\gamma \geq 0$ ($\mu \geq 0, \nu \geq 0$), $\xi \in [0, 1/2]$, $\delta \geq 1$ and $c > -1$. Further let $\beta < 1$ satisfy (2.9), where $\lambda(t)$ is given in (4.1). Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $z^{1-\delta} (\mathcal{B}_c^\delta(f)(z))^\delta$ belongs to the class $\mathcal{S}^*(\xi)$, whenever*

$$c \leq \begin{cases} 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}, & \text{for } \gamma > 0 \text{ } (1 \leq \delta \leq \mu, 1 \leq \delta \leq \nu) \text{ and } \xi \in [0, 1/2]; \\ 3 - \frac{\delta}{\alpha}, & \text{for } \gamma = 0 \text{ } (\mu = 0, \nu = \alpha \in (0, \delta/3] \cup [\delta, \infty)) \text{ and } \xi = 0. \end{cases}$$

Proof. Using $\lambda(t) = (1+c)t^c$, $c > -1$, it can be easily seen that $t\lambda'(t)/\lambda(t) = c$. Applying Theorem 3.1 for $\xi \in (0, 1/2]$, and Theorem 3.2 for $\xi = 0$, the result is immediate. \square

Remark 4.1. 1. For $\delta = 1$ and $\gamma > 0$, [18, Theorem 5.2] (see also [16, Theorem 3.1]) give weaker bounds for c when compared to Theorem 4.1.
2. When $\delta = 1$ and $\xi = 0$, Theorem 4.1 improves the result given in [1, Theorem 5.1] (see also [18, Theorem 5.2]).

Taking $c = 0$ in Theorem 4.1 reduces to the interesting result, which is listed as a corollary.

Corollary 4.1. *Let $\gamma \geq 0$ ($\mu \geq 0, \nu \geq 0$), $\xi \in [0, 1/2]$ and $\delta \geq 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = - \int_0^1 g_{\mu,\nu}^\delta(t) dt,$$

where $g_{\mu,\nu}^\delta(t)$ is given in (2.5) for $\gamma > 0$, and (2.4) for $\gamma = 0$. Moreover, $\mathcal{F}(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re} \left(\left[z \left(\frac{\mathcal{F}(z)}{z} \right)^\delta \right]' + \frac{1}{\delta} \left(\alpha - \gamma \left(1 - \frac{1}{\delta} \right) \right) z \left[z \left(\frac{\mathcal{F}(z)}{z} \right)^\delta \right]'' + \frac{\gamma}{\delta^2} z^2 \left[z \left(\frac{\mathcal{F}(z)}{z} \right)^\delta \right]''' - \beta \right) > 0, \quad (4.2)$$

in the domain \mathbb{D} . Then the function $z^{1-\delta} \mathcal{F}(z)^\delta \in \mathcal{S}^*(\xi)$, whenever

- (i). $\gamma > 0$ ($\mu > 0, \nu > 0$), $\delta \leq \min\{\mu, \nu\}$ and $\xi \in [0, 1/2]$,
- (ii). $\gamma = 0$ ($\mu = 0, \nu = \alpha \geq 0$), $\alpha \in (0, \delta/3] \cup [\delta, \infty)$ and $\xi = 0$.

Proof. It is apparent that for any function $\mathcal{F}(z) \in \mathcal{A}$ satisfying the condition (4.2), then its corresponding function $f(z)$ defined by the relation $(f/z)^\delta = (z(\mathcal{F}/z)^\delta)'$ belongs to $\mathcal{W}_\beta^\delta(\alpha, \gamma)$. Therefore, the integral representation of $\mathcal{F}(z)$ in terms of $f(z)$ is given by

$$\mathcal{F}(z) = \left(\int_0^1 \left(\frac{f(tz)}{t} \right)^\delta dt \right)^{1/\delta}.$$

With the given hypothesis and taking $c = 0$, the result directly follows from Theorem 4.1. \square

Using Corollary 4.1, the following two cases are taken into account:

- (i). Consider $\gamma = 0$, $\delta = 1$, $\alpha \in (0, 1/3] \cup [1, \infty)$ and $\xi = 0$. Let $\beta < 1$ satisfy

$$\frac{\beta}{1-\beta} = - \int_0^1 g_{0,\alpha}^1(t) dt,$$

where $g_{0,\alpha}^1(t)$ is given in (2.4). Using Corollary 4.1, $\operatorname{Re}(\mathcal{F}'(z) + \alpha z \mathcal{F}''(z)) > \beta$, implies $\mathcal{F}(z) \in \mathcal{S}^*$. When $\alpha = 1$,

$$\operatorname{Re}(\mathcal{F}'(z) + z \mathcal{F}''(z)) > \beta = \frac{1 - 2 \ln 2}{2 - 2 \ln 2} \approx -0.62944$$

implies $\mathcal{F}(z) \in \mathcal{S}^*$.

(ii). Consider $\alpha = 3$, $\gamma = 1$, i.e., $(\mu = \nu = 1)$, $\xi \in [0, 1/2]$ and $\delta = 1$. Let $\beta < 1$ satisfy

$$\frac{\beta}{1 - \beta} = - \int_0^1 g_{1,1}^1(t) dt,$$

where $g_{1,1}^1(t)$ is given in (2.5). Further, using the series representation of the function $g_{1,1}^1(t)$ given in (5.10), by a simple computation, we get

$$\frac{\beta}{(1 - \beta)} = 1 - \frac{\pi^2}{6}.$$

Thus, using Corollary 4.1,

$$\operatorname{Re}(\mathcal{F}'(z) + 3z \mathcal{F}''(z) + z^2 \mathcal{F}'''(z)) > \beta = \frac{(\pi^2 - 6)}{(\pi^2 - 12)} \approx -1.81637$$

Remark 4.2. The sharp range for β improves the result obtained in [3, Example 4.4].

Secondly, consider the case when $\gamma = 0$, $\xi \in [0, 1/2]$ and $\lambda(1) = 0$. In order to prove Theorem 2.2 for the given cases, it is enough to show that the function $a(t)$ defined in (3.6) decreases for $t \in (0, 1)$. Now taking the logarithmic derivative of $a(t)$ will give

$$\frac{a'(t)}{a(t)} = \frac{2(t + \xi + \xi t)}{(1 - t^2)b(t)} \left(b(t) + \frac{(1 - t^2)b'(t)}{2(t + \xi + \xi t)} \right).$$

It is easy to see that the terms $(t + \xi + \xi t)$, $(1 - t^2)$, $a(t)$ and $b(t)$ are positive for all values of $t \in (0, 1)$ and $\xi \in [0, 1/2]$. Thus $a'(t) \leq 0$ is equivalent of obtaining $r(t) \leq 0$, where

$$r(t) := b(t) + \frac{(1 - t^2)b'(t)}{2(t + \xi + \xi t)}.$$

Clearly $r(1) = 0$, therefore if $r(t)$ is increasing function of $t \in (0, 1)$ then $a'(t) \leq 0$ which completes the proof. Hence it is enough to prove

$$r'(t) = \frac{(1 + t)}{2(t + \xi(1 + t))^2} \left(\frac{Y(t)}{t} b'(t) + X(t) b''(t) \right) \geq 0$$

or equivalently,

$$s(t) := \frac{Y(t)}{t} b'(t) + X(t) b''(t) \geq 0, \quad (4.3)$$

where $X(t)$ and $Y(t)$ are given in (3.1), $b'(t)$ and $b''(t)$ are given in (3.7) and (3.8), respectively. Substituting the value of $b'(t)$ and $b''(t)$ in (4.3), $s(t)$ is equivalent to

$$s(t) = \left(\frac{\delta}{\alpha} - 1\right) \left(Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right) t^{\delta/\alpha-3} \Lambda_{\alpha}^{\delta}(t) - \left(Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right) t^{-2} \lambda(t) - X(t) t^{-1} \lambda'(t).$$

The hypothesis $\lambda(1) = 0$ directly implies that $s(1) = 0$. If $s(t)$ is decreasing function of $t \in (0, 1)$, clearly means that $a'(t) \leq 0$. Differentiating $s(t)$ with respect to t gives

$$\begin{aligned} s'(t) = & \left(\frac{\delta}{\alpha} - 1\right) \left(\left(\frac{\delta}{\alpha} - 3\right) \left[Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right] + tY'(t) + \left(\frac{\delta}{\alpha} - 2\right) tX'(t) \right) t^{\delta/\alpha-4} \Lambda_{\alpha}^{\delta}(t) \\ & + \left(-\left(\frac{\delta}{\alpha} - 1\right) \left[Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right] - \left[tX'(t) \left(\frac{\delta}{\alpha} - 2\right) + tY'(t)\right] + 2 \left[X(t) \left(\frac{\delta}{\alpha} - 2\right) + Y(t)\right] \right) t^{-3} \lambda(t) \\ & + \left(-\left[Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right] - tX'(t) + X(t)\right) t^{-2} \lambda'(t) - X(t) t^{-1} \lambda''(t). \end{aligned}$$

When $1 \leq \delta \leq \alpha$, then it is easy to see that

$$\left(\frac{\delta}{\alpha} - 1\right) \left(\left(\frac{\delta}{\alpha} - 3\right) \left[Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right] + tY'(t) + \left(\frac{\delta}{\alpha} - 2\right) tX'(t) \right) \leq 0.$$

Thus, to verify that $a'(t) \leq 0$, it is enough to show

$$\begin{aligned} & \left(\left(\frac{\delta}{\alpha} - 1\right) \left[Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right] + \left[tX'(t) \left(\frac{\delta}{\alpha} - 2\right) + tY'(t)\right] - 2 \left[X(t) \left(\frac{\delta}{\alpha} - 2\right) + Y(t)\right] \right) t^{-3} \lambda(t) \\ & - \left(-\left[Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right] - tX'(t) + X(t) \right) t^{-2} \lambda'(t) + X(t) t^{-1} \lambda''(t) \geq 0, \end{aligned} \quad (4.4)$$

for $1 \leq \delta \leq \alpha$.

By a simple calculation, the terms

$$\begin{aligned} & \left(\left(\frac{\delta}{\alpha} - 1\right) \left[Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right] + \left[tX'(t) \left(\frac{\delta}{\alpha} - 2\right) + tY'(t)\right] - 2 \left[X(t) \left(\frac{\delta}{\alpha} - 2\right) + Y(t)\right] \right), \\ \text{and} \quad & \left(-\left[Y(t) + \left(\frac{\delta}{\alpha} - 2\right) X(t)\right] - tX'(t) + X(t) \right) \end{aligned} \quad (4.5)$$

are positive for $1 \leq \delta \leq \alpha$.

Thus, to prove inequality (4.4) for $1 \leq \delta \leq \alpha$, it is enough to show $\lambda(t) \geq 0$, $\lambda'(t) \leq 0$ and $\lambda''(t) \geq 0$.

For the function

$$\omega(1-t) = 1 + \sum_{n=1}^{\infty} x_n (1-t)^n, \quad x_n \geq 0, \quad t \in (0, 1),$$

define

$$\lambda(t) = K t^{b-1} (1-t)^{c-a-b} \omega(1-t), \quad (4.6)$$

where K is chosen such that it satisfies normalization condition $\int_0^1 \lambda(t) dt = 1$. Thus the weighted integral operator defined in (1.2) with $\lambda(t)$ given by (4.6) is represented as

$$H_{a,b,c}^\delta(f)(z) = \left(K \int_0^1 t^{b-1} (1-t)^{c-a-b} \omega(1-t) \left(\frac{f(tz)}{t} \right)^\delta dt \right)^{1/\delta}.$$

This operator is new in the literature whereas for the particular cases of this operator many interesting results are available. For example, when $\delta = 1$, the operator $H_{a,b,c}^1$ was discussed in the literature by several authors for similar problems. For details refer to [1,8] and references therein.

The following result provides the conditions such that $(z^{1-\delta} (H_{a,b,c}^\delta(f)(z))^\delta)$ belongs to the class $\in \mathcal{S}^*(\xi)$.

Theorem 4.2. *Let $\gamma \geq 0$ ($\mu \geq 0, \nu \geq 0$), $\xi \in [0, 1/2]$, $\delta \geq 1$ and $a, b, c > 0$. Let $\beta < 1$ satisfy (2.9), where $\lambda(t)$ is given by (4.6). Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $(z^{1-\delta} (H_{a,b,c}^\delta(f)(z))^\delta)$ belongs to the class $\in \mathcal{S}^*(\xi)$, whenever*

- (i). $(c-a-b) \geq 1$ and $0 < b \leq 1$ for $\gamma = 0$ and $\delta \leq \alpha$,
- (ii). $(c-a-b) \geq 0$ and $0 < b \leq \left(6 - \frac{\delta}{\mu} - \frac{\delta}{\nu}\right)$ for $\gamma > 0$ and $\delta \leq \min\{\mu, \nu\}$.

Proof. Differentiating $\lambda(t)$ defined in (4.6) will give

$$\lambda'(t) = K t^{b-2} (1-t)^{c-a-b-1} \left([(b-1)(1-t) - (c-a-b)t] \omega(1-t) - t(1-t) \omega'(1-t) \right),$$

and

$$\begin{aligned} \lambda''(t) = & K t^{b-3} (1-t)^{c-a-b-2} \left[\left((b-1)(b-2)(1-t)^2 - 2(b-1)(c-a-b)t(1-t) \right. \right. \\ & \left. \left. + (c-a-b)(c-a-b-1)t^2 \right) \omega(1-t) + \left[2(c-a-b)t - 2(b-1)(1-t) \right] t(1-t) \omega'(1-t) \right. \\ & \left. + t^2(1-t)^2 \omega''(1-t) \right]. \end{aligned}$$

It is easy to note that when $(c-a-b) > 0$, then $\lambda(t)$ defined in (4.6) has $\lambda(1) = 0$.

For the case $\gamma = 0$ ($\mu = 0, \nu = \alpha \geq 0$), from Theorem 2.2, we infer that it is enough to prove (4.4). Clearly for $0 \leq \xi \leq 1/2$ and $1 \leq \delta \leq \alpha$, the conditions in (4.5) are satisfied. Hence, it remains to check the validity of $\lambda(t) \geq 0$, $\lambda'(t) \leq 0$ and $\lambda''(t) \geq 0$. However, using the fact that $\omega(1-t)$, $\omega'(1-t)$ and $\omega''(1-t)$ are non-negative for $t \in (0, 1)$ a simple computation gives $\lambda(t) \geq 0$, $\lambda'(t) \leq 0$ and $\lambda''(t) \geq 0$, when $(c-a-b) \geq 1$ and $b \leq 1$.

Now consider the case $\gamma > 0$. Using $\lambda(t)$ given in (4.6), we get

$$\frac{t\lambda'(t)}{\lambda(t)} = (b-1) - (c-a-b) \frac{t}{1-t} - t \frac{\omega'(1-t)}{\omega(1-t)}.$$

Thus the condition (3.5) is true only when

$$(b-1) - (c-a-b) \frac{t}{1-t} - t \frac{\omega'(1-t)}{\omega(1-t)} \leq \left(5 - \frac{\delta}{\mu} - \frac{\delta}{\nu} \right), \quad (4.7)$$

for $1 \leq \delta \leq \min\{\mu, \nu\}$. Since $\omega(1-t)$ and $\omega'(1-t)$ are non-negative on $t \in (0, 1)$, therefore (4.7) is satisfied if

$$(b-1) - (c-a-b)\frac{t}{1-t} \leq \left(5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}\right),$$

which is true whenever $c \geq a+b$ and $b \leq \left(6 - \frac{\delta}{\mu} - \frac{\delta}{\nu}\right)$. Thus, by the given hypothesis and Theorem 2.2, the result follows directly. \square

Remark 4.3. 1. When $\delta = 1$ and $\gamma > 0$, then Theorem 4.2 gives better range for b when compared to [18, Theorem 5.1] (see also [1, Theorem 5.5] for the case $\xi = 0$).
2. When $\delta = 1$ and $\gamma = 0$, Theorem 4.2 cannot be compared with [18, Theorem 5.1]. This is due to the fact that the bound for α is different in both the cases. Also, when $\xi = 0$, due to different range for α , the bounds for a , b and c are different in Theorem 4.2 when compared with [14, Theorem 2.4].

Consider

$$\lambda(t) = \frac{(1+k)^p}{\Gamma(p)} t^k \left(\log \frac{1}{t}\right)^{p-1}, \quad \delta \geq 0 \quad k > -1. \quad (4.8)$$

Then the integral operator (1.2) defined by the above weight function $\lambda(t)$ is the known as generalized Komatu operator denoted by $(F_{k,p}^\delta)$. This integral operator was considered in the work of A. Ebadian [12]. When $\delta = 1$, the operator is reduced to the one introduced by Y. Komatu [15].

Now, we are in the position to state the following result.

Theorem 4.3. Let $\gamma \geq 0$ ($\mu \geq, \nu \geq 0$), $k > -1$, $p \geq 1$, $\xi \in [0, 1/2]$ and $\delta \geq 1$. Let $\beta < 1$ satisfy (2.9), where $\lambda(t)$ is given in (4.8). Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $z^{1-\delta} (F_{k,p}^\delta(f)(z))^\delta$ belongs to the class $\in \mathcal{S}^*(\xi)$, whenever

- (i). $p \geq 2$ and $-1 < k \leq 0$ for $\gamma = 0$ and $\delta \leq \alpha$,
- (ii). $p \geq 1$ and $-1 < k \leq 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}$ for $\gamma > 0$ and $\delta \leq \min\{\mu, \nu\}$.

Proof. Letting $(c-a-b) = p-1$, $b = k+1$ and $\omega(1-t) = \left(\frac{\log(1/t)}{(1-t)}\right)^{p-1}$. Therefore $\lambda(t)$ given in (4.6) can be represented as

$$\lambda(t) = K t^k (1-t)^{p-1} \omega(1-t), \quad \text{where } K = \frac{(1+k)^p}{\Gamma(p)}.$$

Now, by the given hypothesis the result directly follows from Theorem 4.2. \square

Remark 4.4. 1. For $\delta = 1$ and $\gamma > 0$, Theorem 4.3 yield better range for k when compared to [18, Theorem 5.4] (see also [1, Theorem 5.4] for the case $\xi = 0$).
2. For $\delta = 1$ and $\gamma = 0$, Theorem 4.3 cannot be compared with [18, Theorem 5.4]. This is due to the fact that the bound for α is different in both the cases.

Let

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1} (1-t)^{c-a-b} {}_2F_1 \left(\begin{matrix} c-a, & 1-a \\ c-a-b+1 \end{matrix}; 1-t \right),$$

then the integral operator (1.2) defined by the above weight function $\lambda(t)$ is the known as generalized Hohlov operator denoted by $\mathcal{H}_{a,b,c}^\delta$. This integral operator was considered in the work of A. Ebadian [12]. When $\delta = 1$, the reduced integral transform can be represented as the convolution of the normalized hypergeometric function with the analytic

function $z {}_2F_1(a, b; c; z) * f(z)$, which was introduced in the work of Y. C. Kim and F. Ronning [14] and studied by several authors later. The operator $\mathcal{H}_{a,b,c}^\delta$ with $a = 1$ is the generalized Carlson-Shaffer operator ($\mathcal{L}_{b,c}^\delta$) [6].

Using the above operators the following results are obtained.

Theorem 4.4. *Let $\gamma \geq 0$ ($\mu \geq 0, \nu \geq 0$), $\delta \geq 1$, $\xi \in [0, 1/2]$ and $a, b, c > 0$. Let $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \int_0^1 t^{b-1}(1-t)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right) g_{\mu,\nu}^\delta(t) dt, \quad (4.9)$$

where $g_{\mu,\nu}^\delta(t)$ is given in (2.5) for $\gamma > 0$, and (2.4) for $\gamma = 0$. Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $z^{1-\delta} (\mathcal{H}_{a,b,c}^\delta(f)(z))^\delta$ belongs to the class $\mathcal{S}^*(\xi)$, whenever

- (i). $(c-a-b) \geq 1$ and $0 < b \leq 1$ for $\gamma = 0$ and $\delta \leq \alpha$,
- (ii). $(c-a-b) \geq 0$ and $0 < b \leq 6 - \frac{\delta}{\mu} - \frac{\delta}{\nu}$ for $\gamma > 0$ and $\delta \leq \min\{\mu, \nu\}$.

Proof. Choosing

$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \quad \text{and} \quad \omega(1-t) = {}_2F_1\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right),$$

in Theorem 4.2 will give the required result. \square

For $a = 1$, Theorem 4.4 lead to the following corollaries.

Corollary 4.2. *Let $\gamma \geq 0$ ($\mu \geq 0, \nu \geq 0$), $\delta \geq 1$, $\xi \in [0, 1/2]$ and $b, c > 0$. Let $\beta < 1$ satisfy*

$$\frac{\beta}{(1-\beta)} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} g_{\mu,\nu}^\delta(t) dt,$$

where $g_{\mu,\nu}^\delta(t)$ is given in (2.5) for $\gamma > 0$, and (2.4) for $\gamma = 0$. Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $z^{1-\delta} (\mathcal{L}_{b,c}^\delta(f)(z))^\delta$ belongs to the class $\mathcal{S}^*(\xi)$, whenever

- (i). $(c-b) \geq 2$ and $0 < b \leq 1$ for $\gamma = 0$ and $\delta \leq \alpha$,
- (ii). $(c-b) \geq 1$ and $0 < b \leq 6 - \frac{\delta}{\mu} - \frac{\delta}{\nu}$ for $\gamma > 0$ and $\delta \leq \min\{\mu, \nu\}$.

Corollary 4.3. *Consider $\gamma \geq 0$ ($\mu \geq 0, \nu \geq 0$), $b > 0$, $c > 0$ and $\delta \geq 1$. Let $\beta_0 < \beta < 1$, where*

$$\beta_0 = 1 - \frac{1}{2 \left(1 - {}_5F_4 \left(\begin{matrix} 1, b, (2-\xi), \frac{\delta}{\mu}, \frac{\delta}{\nu} \\ c, (1-\xi), \left(1 + \frac{\delta}{\mu}\right), \left(1 + \frac{\delta}{\nu}\right) \end{matrix}; -1 \right) \right)}.$$

Then, for $f \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $\mathcal{L}_{b,c}^\delta(f)(z) \in \mathcal{S}_s^*(\xi)$ or $(z^{1-\delta} (\mathcal{L}_{b,c}^\delta(f)(z))^\delta) \in \mathcal{S}^*(\xi)$, where $\xi = (1 - \delta(1 - \zeta))$, $\xi \in [0, 1/2]$, whenever

- (i). $(c-b) \geq 2$ and $0 < b \leq 1$ for $\gamma = 0$ and $\delta \leq \alpha$,
- (ii). $(c-b) \geq 1$ and $0 < b \leq 6 - \frac{\delta}{\mu} - \frac{\delta}{\nu}$ for $\gamma > 0$ and $\delta \leq \min\{\mu, \nu\}$.

Proof. Letting $a = 1$ in (4.9) and using (5.11) gives

$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left({}_2F_3 \left(\begin{matrix} 1, (2-\xi), \frac{\delta}{\mu}, \frac{\delta}{\nu} \\ (1-\xi), \left(1+\frac{\delta}{\mu}\right), \left(1+\frac{\delta}{\nu}\right) \end{matrix} ; -t \right) - 1 \right) dt,$$

which is equivalent to

$$\frac{\beta - 1/2}{1 - \beta} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} {}_4F_3 \left(\begin{matrix} 1, (2-\xi), \frac{\delta}{\mu}, \frac{\delta}{\nu} \\ (1-\xi), \left(1+\frac{\delta}{\mu}\right), \left(1+\frac{\delta}{\nu}\right) \end{matrix} ; -t \right) dt.$$

Using the series representation of generalized hypergeometric function and applying integration over $t \in (0, 1)$ provides

$$\frac{\beta - 1/2}{1 - \beta} = -{}_5F_4 \left(\begin{matrix} 1, b, (2-\xi), \frac{\delta}{\mu}, \frac{\delta}{\nu} \\ c, (1-\xi), \left(1+\frac{\delta}{\mu}\right), \left(1+\frac{\delta}{\nu}\right) \end{matrix} ; -1 \right).$$

Thus, applying Theorem 4.4 will give the required result. \square

For the two complex parameters $a, b > -1$, consider

$$\lambda(t) = \begin{cases} (a+1)(b+1) \frac{t^a(1-t^{b-a})}{b-a}, & b \neq a, \\ (a+1)^2 t^a \log(1/t), & b = a. \end{cases} \quad (4.10)$$

Then the corresponding integral operator (1.2) obtained using $\lambda(t)$ defined in (4.10) is denoted by $\mathcal{G}_{a,b}^\delta$. This operator is new in the literature whereas for the particular cases of this operator many interesting results are available. For example, when $\delta = 1$, the operator was considered by several authors (see, for example [1, 8, 18]).

Now, using this generalized operator the next result is given.

Theorem 4.5. *Let $a > -1$, $b > -1$, $\gamma \geq 0$ ($\mu \geq 0, \nu \geq 0$), $\xi \in [0, 1/2]$ and $\delta \geq 1$. Let $\beta < 1$ satisfy (2.9), where $\lambda(t)$ is given in (4.10). Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $z^{1-\delta} (\mathcal{G}_{a,b}^\delta(f)(z))^\delta$ belongs to the class $\mathcal{S}^*(\xi)$, whenever*

$$-1 < b = a \leq \begin{cases} 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}, & \gamma > 0 (\mu > 0, \nu > 0), \delta \leq \min\{\mu, \nu\}, \\ 0, & \gamma = 0 (\mu = 0, \nu = \alpha \geq \delta), \end{cases}$$

or

$$-1 < b < a \quad \text{and} \quad \begin{cases} b \in \left[0, \left(5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}\right)\right], & \gamma > 0 (\mu > 0, \nu > 0), \delta \leq \min\{\mu, \nu\}, \\ \gamma = 0 (\mu = 0, \nu = \alpha \geq \delta). \end{cases}$$

Proof. Using $\lambda(t)$ defined by (4.10), we have

$$\frac{t\lambda'(t)}{\lambda(t)} = \begin{cases} \frac{(a - bt^{b-a})}{(1 - t^{b-a})}, & b \neq a, \\ a - \frac{1}{\log(1/t)}, & b = a. \end{cases}$$

Considering both the possibilities, the proof can be divided into the following cases.

Case(i): At first, let $a = b > -1$ and $\gamma > 0$ ($\mu > 0, \nu > 0$). Substituting the values of $t\lambda'(t)/\lambda(t)$ in (3.5) and on further simplification gives

$$a - \frac{1}{\log(1/t)} \leq 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}. \quad (4.11)$$

For $t \in (0, 1)$, the function $\log(1/t)$ is positive. Thus the condition (4.11) is valid, when

$$a \leq 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu} \quad \text{and} \quad 1 \leq \delta \leq \min\{\mu, \nu\}.$$

Now, consider $a = b > -1$ and $\gamma = 0$ ($\mu = 0, \nu = \alpha \geq 0$). For the function $\lambda(t) = (a+1)^2 t^a \log(1/t)$ implies $\lambda(1) = 0$. An easy computation gives

$$\lambda'(t) = (a+1)^2 t^{a-1} (a \log(1/t) - 1) \quad \text{and} \quad \lambda''(t) = (a+1)^2 t^{a-2} (a(a-1) \log(1/t) - (2a-1)).$$

Therefore, for $a \leq 0$, it is easy to see that $\lambda(t) \geq 0$, $\lambda'(t) \leq 0$ and $\lambda''(t) \geq 0$, which implies that the condition (4.4) is true. Hence by given hypothesis and Theorem 2.2, the result directly follows.

Case(ii): Consider $a \neq b$. At first, let $-1 < b < a$ and $\gamma > 0$. Substituting the values of $t\lambda'(t)/\lambda(t)$ in (3.5) is equivalent to the inequality $\phi_t(a) \leq \phi_t(b)$, $t \in (0, 1)$, where

$$\phi_t(b) := bt^b - \left(5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}\right) t^b.$$

Now, it is required to claim that for $b \in [0, (5 - \delta/\mu - \delta/\nu)]$, $\phi_t(b)$ is decreasing function of b . Differentiating $\phi_t(b)$ with respect to b gives

$$\phi'_t(b) := bt^{b-1} \left(b - \left(5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}\right)\right).$$

If $b \leq 0$, then for $1 \leq \delta \leq \mu$ and $1 \leq \delta \leq \nu$, clearly implies that $(b - (5 - \delta/\mu - \delta/\nu)) \leq 0$ which means $\phi'_t(b) \geq 0$. But, it requires to prove that $\phi'_t(b) \leq 0$, therefore we will consider the case only when $b \geq 0$. Thus $\phi'_t(b) \leq 0$ is true for $b \leq (5 - \delta/\mu - \delta/\nu)$. Hence the desired conclusion follows by the given hypothesis and Theorem 3.1.

Now, consider $a \neq b$ and $\gamma = 0$. For the function $\lambda(t) = (a+1)(b+1)t^a(1 - t^{b-a})/b - a$, clearly implies $\lambda(1) = 0$. An easy computation gives

$$\lambda'(t) = (a+1)(b+1) \frac{t^{a-1}(a - bt^{b-a})}{b - a} \quad \text{and} \quad \lambda''(t) = (a+1)(b+1) \frac{t^{a-1}(a(a-1) - b(b-1)t^{b-a})}{b - a}.$$

To prove the required result, it is enough to get the inequality (4.4). Since (4.5) is negative when $1 \leq \delta \leq \alpha$, therefore it remains to prove that $\lambda(t) \geq 0$, $\lambda'(t) \leq 0$ and $\lambda''(t) \geq 0$, which is clearly true when $-1 < a < b$ or $-1 < b < a$ and this completes the proof. \square

Remark 4.5. 1. Consider $a = b$, then for the case $\delta = 1$ and $\gamma > 0$, Theorem 4.5 gives better bound for a as compared to [18, Theorem 5.7] but for $\gamma = 0$, the bound is weaker (see also [1, Theorem 5.3] for the case $\xi = 0$).

2. Consider $b < a$ or $a < b$, then for $\delta = 1$, Theorem 4.5 gives better range for a as compared to [18, Theorem 5.7] (see also [1, Theorem 5.3] for the case $\xi = 0$).

Let

$$\lambda(t) = \frac{(1-k)(3-k)}{2} t^{-k} (1-t^2), \quad 0 \leq k < 1. \quad (4.12)$$

Then the corresponding integral operator (1.2) by taking $\lambda(t)$ given in (4.12) is denoted by $T_k^\delta(f)$. This operator is new in the literature whereas for the particular cases of this operator many interesting results are available. For example, when $\delta = 1$, the operator T_k^1 was considered in the work of R. M. Ali and V. Singh [2].

Now, using this generalized operator the following result is obtained as under.

Theorem 4.6. *Let $\gamma \geq 0$ ($\mu \geq 0, \nu \geq 0$), $k \geq 0$, $\xi \in [0, 1/2]$ and $\delta \geq 1$. Let $\beta < 1$ satisfy (2.9), where $\lambda(t)$ is given in (4.12). Then for $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the function $z^{1-\delta} (T_k^\delta(f)(z))^\delta$ belongs to the class $\mathcal{S}^*(\xi)$, whenever*

- (i). $k \geq 0$, for $\gamma > 0$ ($\delta \leq \mu, \delta \leq \nu$),
- (ii). $k \in [2/3, 1] \cup [3, \infty)$, for $\gamma = 0$ ($\mu = 0, \nu = \alpha(1 \leq \delta \leq \alpha)$).

Proof. Using $\lambda(t)$ defined by (4.12) gives

$$\frac{t\lambda'(t)}{\lambda(t)} = -k - \frac{2t^2}{(1-t^2)}.$$

Now the proof is divided into the following two cases.

Case(i): For $\gamma > 0$. Substituting the values of $t\lambda'(t)/\lambda(t)$ in (3.5) gives

$$-k - \frac{2t^2}{(1-t^2)} \leq 5 - \frac{\delta}{\mu} - \frac{\delta}{\nu}. \quad (4.13)$$

For $k \geq -(5 - \delta/\mu - \delta/\nu)$, (4.13) is obviously true. As we know that $1 \leq \delta \leq \mu$ and $1 \leq \delta \leq \nu$, therefore for all values of $k \geq 0$, (4.13) holds.

Case(ii): Consider $\gamma = 0$. The function $\lambda(t) = (1-k)(3-k)t^{-k}(1-t^2)/2$ implies $\lambda(1) = 0$. Now, in order to prove the result, we need to show that the inequality (4.4) holds under the given hypothesis. Differentiating $\lambda(t)$ gives

$$\begin{aligned} \lambda'(t) &= \frac{(1-k)(3-k)}{2} t^{-k-1} (-k - (2-k)t^2) \quad \text{and} \\ \lambda''(t) &= \frac{(1-k)(3-k)}{2} t^{-k-2} (k(k+1) - (2-k)(1-k)t^2). \end{aligned}$$

For $2/3 \leq k \leq 1$ or $k \geq 3$ gives $\lambda(t) \geq 0$, $\lambda'(t) \leq 0$ and $\lambda''(t) \geq 0$. Thus inequality (4.4) is true for all $k \in [2/3, 1] \cup [3, \infty)$ and $1 \leq \delta \leq \alpha$. \square

5. PROOF OF THEOREM 2.1

To show that with the given conditions

$$z^{1-\delta} (F_\delta(z))^\delta \in \mathcal{S}^*(\xi) \iff \mathcal{N}_{\Pi_{\mu,\nu}^\delta}(h_\xi) \geq 0,$$

it requires to prove that the function $z^{1-\delta} (F_\delta(z))^\delta$ is univalent and satisfies the order of starlikeness condition when $\mathcal{N}_{\Pi_{\mu,\nu}^\delta}(h_\xi) \geq 0$, and its vice versa.

Since the case $\gamma = 0$ ($\mu = 0, \nu = \alpha > 0$) corresponds to [12, Theorem 2.1], it is enough to obtain the condition for $\gamma > 0$. Let

$$H(z) := (1 - \alpha + 2\gamma) \left(\frac{f}{z} \right)^\delta + \left(\alpha - 3\gamma + \gamma \left[\left(1 - \frac{1}{\delta} \right) \left(\frac{zf'}{f} \right) + \frac{1}{\delta} \left(1 + \frac{zf''}{f'} \right) \right] \right) \left(\frac{f}{z} \right)^\delta \left(\frac{zf'}{f} \right). \quad (5.1)$$

Using (2.1) in (5.1) a simple computation gives

$$\begin{aligned} H(z) &= (1 - \mu - \nu + \mu\nu) \left(\frac{f}{z} \right)^\delta + (\mu + \nu - 2\mu\nu) \left(\frac{f}{z} \right)^\delta \left(\frac{zf'}{f} \right) \\ &\quad + \mu\nu \left(\left(1 - \frac{1}{\delta} \right) \left(\frac{zf'}{f} \right) + \frac{1}{\delta} \left(1 + \frac{zf''}{f'} \right) \right) \left(\frac{f}{z} \right)^\delta \left(\frac{zf'}{f} \right) \\ &= \frac{\mu\nu}{\delta^2} z^{1-\delta/\mu} \left(z^{\delta/\mu-\delta/\nu+1} \left(z^{\delta/\nu} \left(\frac{f}{z} \right)^\delta \right)' \right)'. \end{aligned}$$

Let $G(z) = (H(z) - \beta)/(1 - \beta)$, then it is easy to see that for some $\phi \in \mathbb{R}$, $\operatorname{Re}(e^{i\phi} G(z)) > 0$. Now, using duality theory [17, p. 22], we may confine to the function $f(z)$ for which $G(z) = (1 + xz)/(1 + yz)$, where $|x| = |y| = 1$. Thus

$$\frac{\mu\nu}{\delta^2} z^{1-\delta/\mu} \left(z^{\delta/\mu-\delta/\nu+1} \left(z^{\delta/\nu} \left(\frac{f}{z} \right)^\delta \right)' \right)' = (1 - \beta) \frac{1 + xz}{1 + yz} + \beta,$$

or equivalently,

$$\begin{aligned} \left(\frac{f(z)}{z} \right)^\delta &= \frac{\delta^2}{\mu\nu z^{\delta/\nu}} \left(\int_0^z \frac{1}{\eta^{\delta/\mu-\delta/\nu+1}} \left(\int_0^\eta \frac{1}{\omega^{1-\delta/\mu}} \left((1 - \beta) \frac{1 + x\omega}{1 + y\omega} + \beta \right) d\omega \right) d\eta \right) \\ &= \frac{\delta^2}{\mu\nu} \int_0^1 \int_0^1 \left(\beta + (1 - \beta) \frac{1 + xzrs}{1 + yzrs} \right) r^{\delta/\nu-1} s^{\delta/\mu-1} dr ds. \end{aligned}$$

Using (2.2), the above equality implies

$$\left(\frac{f(z)}{z} \right)^\delta = \left(\beta + (1 - \beta) \left(\frac{1 + xz}{1 + yz} \right) \right) * \psi_{\mu,\nu}^\delta(z). \quad (5.2)$$

Therefore

$$\left(z \left(\frac{f(z)}{z} \right)^\delta \right)' = \left(\beta + (1 - \beta) \left(\frac{1 + xz}{1 + yz} \right) \right) * (z \psi_{\mu,\nu}^\delta(z))'.$$

Using (2.3), the above equality can be rewritten as

$$\left(z \left(\frac{f(z)}{z} \right)^\delta \right)' = \left(\beta + (1 - \beta) \left(\frac{1 + xz}{1 + yz} \right) \right) * \Phi_{\mu,\nu}^\delta(z). \quad (5.3)$$

From the integral transform given by (1.2), it is easy to see that

$$\left(\frac{F_\delta(z)}{z} \right)^\delta = \int_0^1 \lambda(t) \left(\frac{f(tz)}{tz} \right)^\delta dt = \int_0^1 \frac{\lambda(t)}{1 - tz} dt * \left(\frac{f(z)}{z} \right)^\delta$$

$$\Longleftrightarrow \left(z \left(\frac{F_\delta(z)}{z} \right)^\delta \right)' = \int_0^1 \frac{\lambda(t)}{1-tz} dt * \left(z \left(\frac{f(z)}{z} \right)^\delta \right)' . \quad (5.4)$$

Using (5.3) in (5.4) gives

$$\begin{aligned} \left(z \left(\frac{F_\delta(z)}{z} \right)^\delta \right)' &= \int_0^1 \frac{\lambda(t)}{1-tz} dt * (1-\beta) \left(\frac{\beta}{(1-\beta)} + \frac{1+xz}{1+yz} \right) * \Phi_{\mu,\nu}^\delta(z) \\ \Longleftrightarrow \left(z \left(\frac{F_\delta(z)}{z} \right)^\delta \right)' &= (1-\beta) \left(\int_0^1 \lambda(t) \Phi_{\mu,\nu}^\delta(tz) dt + \frac{\beta}{(1-\beta)} \right) * \left(\frac{1+xz}{1+yz} \right) . \end{aligned}$$

By Noshiro-Warschawski's Theorem (for details see [11, Theorem 2.16]), the function $z^{1-\delta}(F_\delta(z))^\delta$ defined in the unit disk \mathbb{D} belongs to \mathcal{S} if $(z^{1-\delta}(F_\delta(z))^\delta)'$ is contained in the half plane not containing the origin.

From the result given in [17, P. 23] and by the above equality, it is easy to note that

$$\begin{aligned} 0 \neq \left(z (F_\delta/z)^\delta \right)' &\Longleftrightarrow \operatorname{Re} (1-\beta) \left(\int_0^1 \lambda(t) \Phi_{\mu,\nu}^\delta(tz) dt + \frac{\beta}{(1-\beta)} \right) > \frac{1}{2} \\ &\Longleftrightarrow \operatorname{Re} (1-\beta) \left(\int_0^1 \lambda(t) \Phi_{\mu,\nu}^\delta(tz) dt + \frac{\beta}{(1-\beta)} - \frac{1}{2(1-\beta)} \right) > 0. \end{aligned}$$

Using (2.9) in the above inequality implies

$$\operatorname{Re} \int_0^1 \lambda(t) \left(\Phi_{\mu,\nu}^\delta(tz) - \frac{1+g_{\mu,\nu}^\delta(t)}{2} \right) dt > 0. \quad (5.5)$$

Further using (2.3) and (2.6) in (5.5) gives

$$\operatorname{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{r^{\delta/\nu-1} s^{\delta/\mu-1}}{(1-rstz)^2} dr ds - \int_0^1 \int_0^1 \frac{1-\xi(1+rst)}{(1-\xi)(1+rst)^2} r^{\delta/\nu-1} s^{\delta/\mu-1} dr ds \right) dt > 0.$$

Evidently $\operatorname{Re} \left(\frac{1}{1-rstz} \right)^2 \geq \frac{1}{(1+rst)^2}$ for $z \in \mathbb{D}$, directly implies that

$$\begin{aligned} \operatorname{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{r^{\delta/\nu-1} s^{\delta/\mu-1}}{(1-rstz)^2} dr ds - \int_0^1 \int_0^1 \frac{1-\xi(1+rst)}{(1-\xi)(1+rst)^2} r^{\delta/\nu-1} s^{\delta/\mu-1} dr ds \right) dt \\ \geq \frac{\xi}{(1-\xi)} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{(trs) r^{\delta/\nu-1} s^{\delta/\mu-1}}{(1+rst)^2} dr ds \right) dt > 0. \end{aligned}$$

Clearly $f_1(r) = (trs)r^{\delta/\nu-1}/(1+rst)^2$ is positive function of r satisfying $0 < r \ll 1$ whenever $0 < s < 1$ and $0 < t < 1$. Hence the integral of $f_1(r)$ with respect to r yields positive values that lie over r axis, which implies that the function $f_2(s) = s^{\delta/\mu-1}(\int_0^1 f_1(r) dr)$ is positive for $s \in (0, 1)$ and hence the integral of function $f_2(s)$ is positive above s -axis over $s \in (0, 1)$. Since the function $\lambda(t)$ is non-negative for $t \in (0, 1)$, the function $\int_0^1 \lambda(t)(\int_0^1 f_2(s) ds) dt \geq 0$. Thus, $\operatorname{Re}(z^{1-\delta}(F_\delta(z))^\delta)' > 0$ which leads to the conclusion that the function $(z^{1-\delta}(F_\delta(z))^\delta)$ is univalent in \mathbb{D} .

In the next part of the theorem the condition of starlikeness is discussed for $\gamma > 0$ ($\mu > 0, \nu > 0$). From the theory of convolution [17, P. 94], it is clear that

$$g \in \mathcal{S}^*(\xi) \iff \frac{1}{z}(g * h_\xi)(z) \neq 0, \quad (5.6)$$

where $h_\xi(z)$ is as defined in (2.10).

For $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, the weighted integral operator F_δ defined in (1.2), belong to the class $\mathcal{S}_s^*(\zeta)$ with the conditions $(1 - \frac{1}{\delta}) \leq \zeta \leq (1 - \frac{1}{2\delta})$ and $\delta \geq 1$, is equivalent of obtaining $z \left(\frac{F_\delta}{z}\right)^\delta \in \mathcal{S}^*(\xi)$, where $\xi = 1 - \delta + \delta\zeta$ and $0 \leq \xi \leq \frac{1}{2}$, i.e.,

$$F_\delta \in \mathcal{S}_s^*(\zeta) \iff z \left(\frac{F_\delta}{z}\right)^\delta \in \mathcal{S}^*(\xi) \quad (5.7)$$

with the above conditions. From (5.6), it clearly follows that

$$z \left(\frac{F_\delta}{z}\right)^\delta \in \mathcal{S}^*(\xi) \iff 0 \neq \frac{1}{z} \left(z \left(\frac{F_\delta}{z}\right)^\delta * h_\xi(z) \right).$$

Using (5.4), the above inequality reduces to its equivalent form as

$$0 \neq \int_0^1 \lambda(t) \left(\frac{f(tz)}{tz} \right)^\delta dt * \frac{h_\xi(z)}{z} = \int_0^1 \frac{\lambda(t)}{1-tz} dt * \left(\frac{f(z)}{z} \right)^\delta * \frac{h_\xi(z)}{z}.$$

From equation (5.2), substituting the value of $(f/z)^\delta$ in the above inequality leads to

$$0 \neq (1-\beta) \left(\left[\int_0^1 \lambda(t) \frac{h_\xi(tz)}{tz} dt + \frac{\beta}{(1-\beta)} \right] * \psi_{\mu,\nu}^\delta(z) * \frac{1+xz}{1+yz} \right)$$

which clearly holds if, and only if,

$$\operatorname{Re} (1-\beta) \left(\int_0^1 \lambda(t) \frac{h_\xi(tz)}{tz} dt + \frac{\beta}{(1-\beta)} \right) * \psi_{\mu,\nu}^\delta(z) > \frac{1}{2}.$$

Using (2.9), the above expression gives

$$\operatorname{Re} \int_0^1 \lambda(t) \left(\frac{h_\xi(tz)}{tz} - \frac{1+g_{\mu,\nu}^\delta(t)}{2} \right) dt * \psi_{\mu,\nu}^\delta(z) \geq 0$$

which on further using (2.2) leads to

$$\operatorname{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{h_\xi(tz r^{\nu/\delta} s^{\mu/\delta})}{tz r^{\nu/\delta} s^{\mu/\delta}} dr ds - \frac{1+g_{\mu,\nu}^\delta(t)}{2} \right) dt \geq 0$$

or equivalently,

$$\operatorname{Re} \int_0^1 \lambda(t) \left(\frac{\delta^2}{\mu\nu} \int_0^1 \int_0^1 \frac{h_\xi(tz uv)}{tz uv} u^{\delta/\nu-1} v^{\delta/\mu-1} du dv - \frac{1+g_{\mu,\nu}^\delta(t)}{2} \right) dt \geq 0.$$

Moreover, changing the variable $tu = \omega$ gives

$$\operatorname{Re} \int_0^1 \frac{\lambda(t)}{t^{\delta/\nu}} \left(\frac{\delta^2}{\mu\nu} \int_0^t \int_0^1 \frac{h_\xi(\omega zv)}{\omega zv} \omega^{\delta/\nu-1} v^{\delta/\mu-1} d\omega dv - t^{\delta/\nu} \frac{1+g_{\mu,\nu}^\delta(t)}{2} \right) dt \geq 0.$$

Now, integrating the above expression with respect to t and using (2.7) leads to

$$\operatorname{Re} \int_0^1 \Lambda_\nu^\delta(t) \frac{d}{dt} \left(\frac{\delta^2}{\mu\nu} \int_0^t \int_0^1 \frac{h_\xi(\omega zv)}{\omega zv} \omega^{\delta/\nu-1} v^{\delta/\mu-1} d\omega dv - t^{\delta/\nu} \frac{1 + g_{\mu,\nu}^\delta(t)}{2} \right) dt \geq 0,$$

which on further using (2.5) gives

$$\operatorname{Re} \int_0^1 \Lambda_\nu^\delta(t) t^{\delta/\nu-1} \left(\int_0^1 \left(\frac{h_\xi(tzv)}{t zv} - \frac{1 - \xi(1 + vt)}{(1 - \xi)(1 + vt)^2} \right) v^{\delta/\mu-1} dv \right) dt \geq 0.$$

Changing the variable $tv = \eta$ gives

$$\operatorname{Re} \int_0^1 \Lambda_\nu^\delta(t) t^{\delta/\nu-\delta/\mu-1} \left(\int_0^t \left(\frac{h_\xi(\eta z)}{\eta z} - \frac{1 - \xi(1 + \eta)}{(1 - \xi)(1 + \eta)^2} \right) \eta^{\delta/\mu-1} d\eta \right) dt \geq 0.$$

Now, integrating with respect to t and using (2.8), finally gives

$$\operatorname{Re} \int_0^1 \Pi_{\mu,\nu}^\delta(t) \frac{d}{dt} \left(\int_0^t \left(\frac{h_\xi(\eta z)}{\eta z} - \frac{1 - \xi(1 + \eta)}{(1 - \xi)(1 + \eta)^2} \right) \eta^{\delta/\mu-1} d\eta \right) dt \geq 0$$

or equivalently,

$$\operatorname{Re} \int_0^1 \Pi_{\mu,\nu}^\delta(t) t^{\delta/\mu-1} \left(\frac{h_\xi(tz)}{tz} - \frac{1 - \xi(1 + t)}{(1 - \xi)(1 + t)^2} \right) dt \geq 0,$$

which means that $\mathcal{N}_{\Pi_{\mu,\nu}^\delta}(h_\xi) \geq 0$ and this completes the proof.

Now, to verify the sharpness let $f(z) \in \mathcal{W}_\beta^\delta(\alpha, \gamma)$, therefore it satisfies the differential equation

$$\frac{\mu\nu}{\delta^2} z^{1-\delta/\mu} \left(z^{\delta/\mu-\delta/\nu+1} \left(z^{\delta/\nu} \left(\frac{f}{z} \right)^\delta \right)' \right)' = \beta + (1 - \beta) \frac{1 + z}{1 - z} \quad (5.8)$$

with $\beta < 1$ defined in (2.9). From (5.8), an easy computation gives

$$z \left(\frac{f}{z} \right)^\delta = z + 2(1 - \beta) \sum_{n=1}^{\infty} \frac{\delta^2 z^{n+1}}{(\delta + n\nu)(\delta + n\mu)}.$$

Using the above expression, (1.2) gives

$$z \left(\frac{F_\delta(z)}{z} \right)^\delta = z \int_0^1 \lambda(t) \left(\frac{f(tz)}{tz} \right)^\delta dt = z + 2(1 - \beta) \sum_{n=1}^{\infty} \frac{\delta^2 \tau_n z^{n+1}}{(\delta + n\nu)(\delta + n\mu)} \quad (5.9)$$

where $\tau_n = \int_0^1 t^n \lambda(t) dt$.

The function $g_{\mu,\nu}^\delta$ defined in (2.6) has its series expansion as

$$g_{\mu,\nu}^\delta(t) = 1 + \frac{2\delta^2}{(1 - \xi)} \sum_{n=1}^{\infty} \frac{(-1)^n (n + 1 - \xi) t^n}{(n\nu + \delta)(n\mu + \delta)}, \quad (5.10)$$

which can further be represented in the form of generalized hypergeometric function as

$$g_{\mu,\nu}^\delta(t) = 2 {}_4F_3 \left(1, (2-\xi), \frac{\delta}{\mu}, \frac{\delta}{\nu}; (1-\xi), \left(1+\frac{\delta}{\mu}\right), \left(1+\frac{\delta}{\nu}\right); -t \right) - 1. \quad (5.11)$$

Now using (5.10) in (2.9) gives

$$\frac{\beta}{(1-\beta)} = -1 - \frac{2\delta^2}{(1-\xi)} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1-\xi) \tau^n}{(n\nu+\delta)(n\mu+\delta)}. \quad (5.12)$$

From (5.9), it is easy to see that

$$\left(z \left(\frac{F_\delta(z)}{z} \right)^\delta \right)' = 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(n+1)\delta^2 \tau_n z^n}{(\delta+n\nu)(\delta+n\mu)},$$

which means that

$$\begin{aligned} z \left(z \left(\frac{F_\delta(z)}{z} \right)^\delta \right)' \Big|_{z=-1} &= -1 + 2\delta^2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n+1-\xi) \tau_n}{(\delta+n\nu)(\delta+n\mu)} \\ &\quad + 2\xi\delta^2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \tau_n}{(\delta+n\nu)(\delta+n\mu)}. \end{aligned}$$

From (5.12), an easy computation gives

$$z \left(z \left(\frac{F_\delta(z)}{z} \right)^\delta \right)' \Big|_{z=-1} = -\xi + 2\xi\delta^2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \tau_n}{(\delta+n\nu)(\delta+n\mu)}.$$

Further using (5.9), the above expression is equivalent to

$$z \left(z \left(\frac{F_\delta(z)}{z} \right)^\delta \right)' \Big|_{z=-1} = \xi \left. z \left(\frac{F_\delta(z)}{z} \right)^\delta \right|_{z=-1}$$

which clearly implies the sharpness of the result.

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